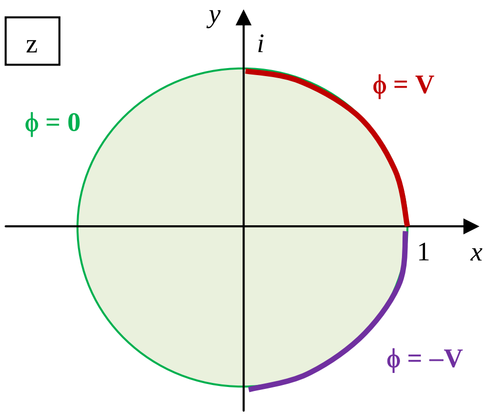
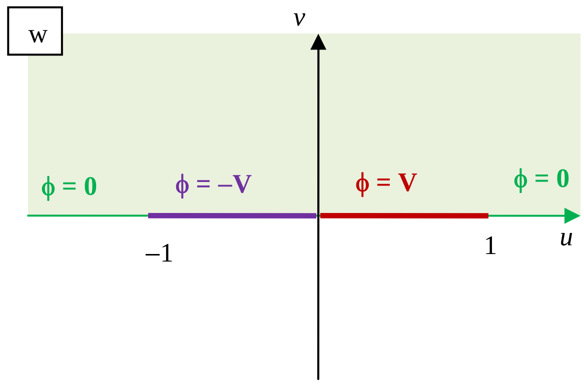
Math 756 • Final Exam • Victor Matveev • December 2013

1) **(25pts)** Solve the Laplace's equation inside a circle with piece-wise constant boundary conditions shown below, by mapping it onto a half-plane. Organize your solution into the following steps:





a) Use cross-ratios to map the circle onto upper half-plane, preserving the boundary conditions, as shown in the picture (hint: to simplify algebra, one of the z_i values should map to ∞). Invert the transform, obtaining both z(w) and w(z).

b) Solve the Laplace's problem in the upper half-plane, expressing the solution in terms of w (hint: $\phi = A \arg(w-1) + B \arg(w) \pm A \arg(w+1)$)

c) Apply the mapping w(z) to the solution to determine the solution inside the circle, expressing it as a function of z (do not combine the three argument terms together).

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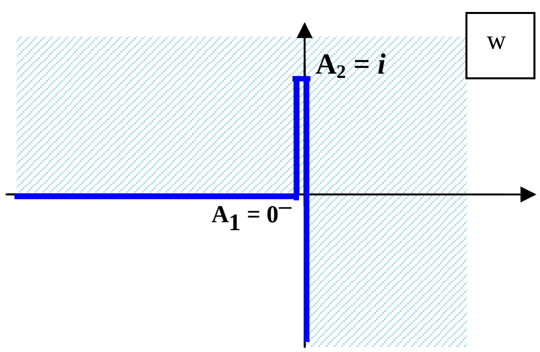
2) **(20pts)** Use the Schwartz-Christoffel transformation to obtain the mapping $w=f(z)$ from the upper half-plane to the following domain (note the obstacle of height 1 along the imaginary axis). To do this, you need to map only two points: $a_1=0$ to $A_1=0^-$, and $a_2=1$ to $A_2=i$



Hint: this map has the form $w(z) = \gamma z^p (z+c) + A$, where p, γ, c and A are some constants.

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3) **(30pts)** Consider a **clockwise** vortex of strength Γ centered at $x_o=2$, near a cylindrical obstacle of radius 1 centered at the origin (as usual, assume ideal fluid flow):

a) Use the Milne-Thompson Theorem to find the complex potential of this flow, and express the resulting complex potential as a superposition of three elementary vortices, indicating the center of each vortex in a sketch (recall that $g(z) = g_o(z) + \overline{g_o(a^2/\bar{z})}$)

b) Find the complex velocity, and the two stagnation points; show them in your sketch. Make a rough sketch of a couple streamlines on either side of the obstacle (but do not solve the streamline equation Im g=const), close to the obstacle surface.

c) Use the Blasius Theorem to find the force exerted on the cylinder by this flow:

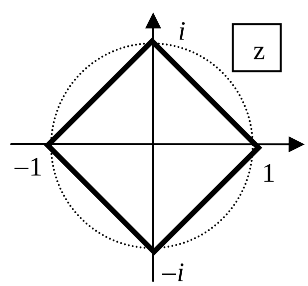
$$\bar{F} = \frac{i\rho}{2} \oint_{\partial B} \left(\frac{dg}{dz} \right)^2 dz \, .$$

4) **(30pts)** Make a rough sketch of the image of a square inscribed into a unit circle (shown below) under the Joukowski transformation w = z + 1 / z. Do not derive the expressions for the image of each side of the square (that's messy!) Instead, perform the following steps:

i) Map the vertices of the square and the point in the middle of one side of the square. Knowledge of the general properties of a conformal map are sufficient to guess the sketch everywhere except near the two non-conformal points.

ii) This map is differentiable for all $z \neq 0$, so you can expand w(z) in a Taylor series to leading order near the two non-conformal points, obtaining $W \approx \gamma Z^n$, where $W = w - w_o$ and $Z = z - z_o$. Thus, near the non-conformal points this map behaves like a simple power function. This will allow you to determine all the angles of intersection of the images of the sides of the square (show these angles clearly in your sketch).

iii) In your sketch, indicate the image of the interior of the square, and the region enclosed between the square and the unit circle.



5) **(15pts extra credit)** Make a rough sketch of the image of the square shown below under the Joukowski transformation w = z + 1/z (note that this is slightly easier than problem 4).